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Alexander Ziwet

LESSONS

ON

HIGHER ALGEBRA

WITH AN APPENDIX ON

THE NATURE OF MATHEMATICAL REASONING

BY

ELLEN HAYES

PROFESSOR OF MATHEMATICS IN WELLESLEY COLLEGE

REVISED EDITION

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Prof. A Ziwet
gt
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MATHEMATICS will ever remain the most perfect type of the Deductive Method in general; and the applications of mathematics to the deductive branches of physics furnish the only school in which philosophers can effectually learn the most difficult and important portion of their art, the employment of the laws of simpler phenomena for explaining and predicting those of the more complex. These grounds are quite sufficient for deeming mathematical training an indispensable basis of real scientific education, and regarding (according to the *dictum* which an old but unauthentic tradition ascribes to Plato) one who is ἀγεωμέτρητος, as wanting in one of the most essential qualifications for the successful cultivation of the higher branches of philosophy. — JOHN STUART MILL: *System of Logic.*

Je me plaisais surtout aux mathématiques, à cause de la certitude et de l'évidence de leurs raisons; mais je ne remarquais point encore leur vrai usage, et, pensant qu'elles ne servaient qu'aux arts mécaniques, je m'etonnais de ce que, leur fondements étant si fermes et si solides, on n'avait rien bâti dessus de plus relevé. — DESCARTES: *Discours sur la Méthode*, 1637.

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PREFACE TO THE FIRST EDITION.

THE following Lessons have been prepared for use in a brief course in Algebra offered in the first part of the second semester, freshman year, at Wellesley College. An effort has been made to give unity to the work by proposing one general problem, — the discovery of the roots of Higher Numerical Equations. Everything has been omitted which does not bear directly on the solution of this problem. It hardly need be added that in so brief a presentation of the subject only the more important theorems could be included. The essentials of the Theory of Logarithms have been presented, preliminary to the study of Trigonometry.

The works chiefly consulted in writing the Lessons, and to which special indebtedness should be acknowledged, are the treatises of Todhunter, Hall and Knight, Chrystal, Burnside and Panton, Serret, and Bertrand.

E. H.

PREFACE TO THE REVISED EDITION.

IN revising the *Lessons on Higher Algebra* various minor changes have been made in the articles of the first edition, and a number of articles added. A section on Determinants has also been introduced. In explanation of the brevity of these Lessons, it ought to be said, that the mathematical course for which they were written includes further work for more advanced students, in which the text-book used is Burnside and Panton's *Theory of Equations.*

The presence, in an algebra book, of so unusual a feature as a section on the nature of mathematical reasoning, calls for more than a passing word.

It cannot have escaped the notice of those who are acquainted with the prevailing methods of instruction in mathematics and logic that, on the one hand, the average teacher of mathematics gives little or no attention to the nature of the processes according to which mathematical reasoning is conducted; while, on the other hand, the teacher of logic almost ignores mathematics as a source of illustrations of the principles with which he is dealing. A preface is not the place to discuss the causes or effects of this divorce of two subjects which might go hand in hand; but I must express my conviction that when we discover those methods of instruction which are essential to sound education, they will be found to include logic for the student of mathematics, and mathematics for the student of logic. Since writing the Appendix I have received the Report of the Committee on Secondary School Studies, appointed at the meeting of the National Educational Association, July 9, 1892, including the Conference Report of the Committee on Mathematics, of which Professor Simon Newcomb, of the Johns Hopkins University, is chairman. This Conference Report confirms me in the belief that any successful effort to improve the teaching of mathematics, both in colleges and secondary schools, must recognize the claims of logic. "The very fact that demonstrative geometry is the most elaborate illustration of formal logic in the entire curriculum of the student, makes the consideration of these elementary principles of logic more interesting and profitable in this connection than in any other." (Report: Mathematics, p. 115.) The analytical branch of mathematics, however, scarcely falls behind the geometrical in affording 'elaborate illustration of formal logic'; and the identity of the logic of algebra and the logic of geometry is probably best shown by putting algebraic and geometrical illustrations side by side.

The closing section of this book does not undertake to give any formal or complete presentation of the logic of mathematics; much less does it furnish those outlines of ratiocination which are essential to a proper understanding of mathematical reasoning. As far as possible, technical terms have been avoided. There is no mention of 'contraposition,' or an 'undistributed middle term,' or an 'illicit process.' These and like matters the individual teacher will introduce or omit according to his own judgment. I have only insisted on the necessity of distinguishing between the conditional proposition of mathematics and the causal proposition of inductive science; and I have emphasized the fact of the constant recurrence of the distributed predicate — or equal quantification of the terms P and Q — in the mathematical proposition: All P is Q.

As regards methods of presenting this proposed combination of logic and mathematics, much might be said; but I may here only suggest that good reasons can be advanced for viewing the "Science of Proof, or Evidence," itself as an observational inductive science, and teaching it as such. It would obviously be absurd to expect students to re-evolve the Systems of Aristotle and Mill, — as absurd as to expect them to re-discover the laws of physics or biology; but, on the other hand, it is not necessary to teach logic after the all too common dogmatic fashion. For instance, one might first allow a student to chance upon the fallacy of the illicit process of the major term, and then encourage him to set to work to determine wherein lies the error. After he has discovered that cases following a certain type are invariably incorrect, it will be time enough to give him the technical term for what he has observed.

In conclusion, whatever may be its faults, the object of the Appendix will be secured if those who read it are stimulated to study formal logic for the purposes of mathematics, and are thence led to explore that larger and more important realm known as Inductive Logic, the principles of which are of daily concern in the pursuit of science and in the conduct of life.

ELLEN HAYES.

WELLESLEY, MASS.,
Feb. 1, 1894.

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LESSONS ON HIGHER ALGEBRA.

I. INTRODUCTORY.

1. When one quantity depends on another for its value, the former is said to be a function of the latter.

E.g. $2\pi r$ is a function of r [illegible] depends on the value of r.

If x represents the fundamental quantity, the usual symbol for function of x is $f(x)$; [illegible] to distinguish one function from another [illegible] $F(x)$, $f_1(x)$, etc.

If the fundamental quantity [illegible] continuously in value, [illegible] quantity is supposed not to vary, [illegible] An arbitrary constant [illegible] trarily assigned. [illegible] does not admit of an arbitrary [illegible] if $f(x)=\frac{[illegible]}{[illegible]}$, [illegible] viewed as a variable, while [illegible] trary value which [illegible] be independent of x.

Algebraic functions [illegible] six operations: [illegible] division, involution, and evolution [illegible] cating the last two operations [illegible] functions are classed as *Transcendental.* With the latter the present Lessons are not concerned.

A *rational* algebraic function of a quantity is one in which the quantity is free from fractional exponents.

An *integral* algebraic function is one in which the quantity is free from negative exponents.

The expression

$$p_0x^n + p_1x^{n-1} + p_2x^{n-2} + \cdots + p_{n-1}x + p_n$$

is a rational integral algebraic function of x, n being positive and integral, and the coefficients $p_0, p_1, \cdots p_n$, being independent of x. The limitations in regard to the exponents are to be understood to apply only to the exponents of x; the coefficients p_0, p_1, etc., may have negative and fractional exponents. Whenever the above expression is used in the following pages, n is positive and integral unless the contrary is expressly stated; also p_0, p_1, etc., are real quantities.

2. As x in $f(x)$ changes, it is obvious that the value of $f(x)$ must change also, whatever kind of function it may be. For example, let $f(x) = \frac{4}{3}\pi x^3$; if we assign the value 3 inches to x, the volume of the sphere is 36π cubic inches; if the radius is 4 inches, the volume is $85\frac{1}{3}\pi$ cubic inches.

Suppose x to take an increment h, so that $f(x)$ becomes $f(x+h)$. If the first value of the function be subtracted from the second, the remainder $f(x+h) - f(x)$ is the increment of the function due to the increment of the variable, and $\frac{f(x+h) - f(x)}{h}$ is the ratio of the increment of the function to that of the variable.

Let h now be supposed to diminish without limit, *i.e.* to *decrease* until it differs from zero by less than any assign-able quantity. The value of the ratio $\frac{f(x+h)-f(x)}{h}$, when this supposition is made regarding h, is represented by $f'(x)$, and is called the *first derived function of* $f(x)$ (or, briefly, the *first derivative*), since it is derived from $f(x)$, and is itself, in general, some function of x. When h diminishes without limit, $f(x+h)-f(x)$ also diminishes without limit, so that in $f'(x)$ we have the ratio of two infinitely small quantities; but the ratio itself is, in general, some finite quantity.

In case the function is such that any real finite value of x renders $f(x)$ imaginary or impossible, $f'(x)$ also becomes imaginary or impossible. For such a value of x, $f(x)$ is said to be discontinuous; see Art. 11.

To illustrate the nature of $\frac{f(x+h)-f(x)}{h}$,

let $$f(x)=3x^2.$$

Then we have, when x takes an increment,

$$f(x+h)=3(x+h)^2=3(x^2+2xh+h^2),$$

and

$$\frac{f(x+h)-f(x)}{h}=\frac{3(x^2+2xh+h^2)-3x^2}{h}=6x+3h;$$

but $6x+3h=6x$ when h is made infinitely small.

$$\therefore\ f'(x)=6x.$$

3. *Given* $\mathrm{f}(\mathrm{x})=\mathrm{f}_1(\mathrm{x})+\mathrm{f}_2(\mathrm{x})$, *to find* $\mathrm{f}'(\mathrm{x})$.

$$\frac{f(x+h)-f(x)}{h}=\frac{f_1(x+h)+f_2(x+h)-f_1(x)-f_2(x)}{h}$$

$$=\frac{f_1(x+h)-f_1(x)}{h}+\frac{f_2(x+h)-f_2(x)}{h}.$$

Therefore in the limit

$$f'(x) = f_1'(x) + f_2'(x); \tag{1}$$

i.e. *the derivative of the sum of two functions is the sum of the derivatives of the functions.*

It is evident that the same proof would apply to any number of functions connected by plus and minus signs.

The derivative of a constant is zero, being the increment of a constant divided by the increment of a variable.

4. *Given* $f(x) = f_1(x)f_2(x)$, *to find* $f'(x)$.

Let $u = f_1(x)$, and $u + h_1 = f_1(x+h)$, h_1 being the increment of u due to the increment h, which x has taken; also let $v = f_2(x)$, and $v + h_2 = f_2(x+h)$.

Then $f(x+h) = f_1(x+h)f_2(x+h) = (u+h_1)(v+h_2)$,

and

$$\begin{aligned} f(x+h) - f(x) &= (u+h_1)(v+h_2) - uv \\ &= vh_1 + uh_2 + h_1h_2; \end{aligned}$$

hence

$$\frac{f(x+h)-f(x)}{h} = \frac{vh_1}{h} + \frac{uh_2}{h} + \frac{h_1h_2}{h};$$

but when h diminishes without limit,

$$\frac{h_1}{h} = f_1'(x), \text{ and } \frac{h_2}{h} = f_2'(x),$$

and the term $\frac{h_1h_2}{h}$ vanishes.

Hence we have

$$\begin{aligned} f'(x) &= vf_1'(x) + uf_2'(x) \\ &= f_2(x)f_1'(x) + f_1(x)f_2'(x), \end{aligned} \tag{2}$$

a result which may be expressed as a theorem. In a similar manner, we may find the derivative of the product of three or more functions of x. Thus if

$$f(x)=f_1(x)f_2(x)f_3(x),$$
$$f'(x)=f_2(x)f_3(x)f_1'(x)+f_1(x)f_3(x)f_2'(x)+f_1(x)f_2(x)f_3'(x).$$

5. In general, let

$$f(x)=f_1(x)f_2(x)\cdots f_n(x);$$

then

$$\begin{aligned}f'(x)=&f_1'(x)f_2(x)\cdots f_n(x)\\&+f_2'(x)f_1(x)f_3(x)\cdots f_n(x)+f_3'(x)f_1(x)f_2(x)\cdots f_n(x)\\&+\cdots+f_n'(x)f_1(x)f_2(x)\cdots f_{n-1}(x),\end{aligned}\tag{3}$$

a polynomial of n terms.

Now suppose

$$f_1(x)=f_2(x)=\cdots f_n(x)=\phi(x);$$

then $\quad f(x)=[\phi(x)]^n;$

and (3) becomes

$$f'(x)=n[\phi(x)]^{n-1}\phi'(x).\tag{4}$$

Hence (4) affords a rule for finding the derivative of a function affected with a positive integral exponent. We notice one special case: when $\phi(x)=x$. The derivative of the fundamental variable itself is seen to be unity; hence $\phi'(x)=1$, and (4) becomes

$$f'(x)=nx^{n-1}.\tag{5}$$

6. *Given* $\mathrm{f}(\mathrm{x})=[\phi(\mathrm{x})]^{\frac{p}{q}}$, *where* p *and* q *are positive and integral; to find* $\mathrm{f}'(\mathrm{x})$.

Raising both members of the given equation to the qth power,

$$[f(x)]^q=[\phi(x)]^p.$$

Since the derivatives of equal functions must be equal, we have by (4)

$$q[f(x)^{q-1}f'(x)=p[\phi(x)]^{p-1}\phi'(x);$$

hence $$f'(x)=\frac{p}{q}\frac{[\phi(x)]^{p-1}\phi'(x)}{f(x)^{q-1}};$$

but $$[f(x)]^{q-1}=[\phi(x)]^{\frac{p}{q}(q-1)};$$

substituting this value in the preceding equation,

$$f'(x)=\frac{p}{q}[\phi(x)]^{\frac{p}{q}-1}\,\phi'(x). \qquad (6)$$

7. *Given* $f(x)=[\phi(x)]^{-n}$, *where* n *is integral; to find* $f'(x)$.

$$f(x)=\frac{1}{[\phi(x)]^n},$$

or $$f(x)[\phi(x)]^n=1.$$

Applying the rule for the derivative of the product of two functions, together with the rule for the derivative of a power of a function, and remembering that the derivative of a constant is zero, we have

$$f'(x)[\phi(x)]^n+nf(x)[\phi(x)]^{n-1}\phi'(x)=0;$$

solving for $f'(x)$, and eliminating $f(x)$ by means of the given equation,

$$f'(x)=-n[\phi(x)]^{n-1}\phi'(x). \qquad (7)$$

8. *Given* $\mathrm{f(x)}=[\phi(\mathrm{x})]^{-\frac{\mathrm{p}}{\mathrm{q}}}$, *where* $-\frac{\mathrm{p}}{\mathrm{q}}$ *indicates a negative fractional exponent; to find* $\mathrm{f'(x)}$.

The equation may be written

$$f(x)[\phi(x)]^{\frac{p}{q}}=1,$$

or

$$[f(x)]^q[\phi(x)]^p=1;$$

and proceeding as in the other cases, we find

$$f'(x)=-\frac{p}{q}[\phi(x)]^{-\frac{p}{q}-1}\phi'(x). \tag{8}$$

Comparing formulas (4), (6), (7), (8), it will be seen that if

$$f(x)=[\phi(x)]^n,$$
$$f'(x)=n[\phi(x)]^{n-1}\phi'(x)$$

for all cases; *i.e.* when n is positive and integral, positive and fractional, negative and integral, negative and fractional.

EXAMPLES.

Find the first derivatives of the following functions:

$$2a^3x;\quad -\tfrac{1}{5}x^5;\quad \pi x^2;\quad 3x^4-cx^2+1;\quad ax^5+bx^4+c;$$

$$-\tfrac{1}{2}x^3+\tfrac{2}{3}x^0;\quad ax^{\frac{2}{3}};\quad \frac{3x^{\frac{2}{3}}}{x^{\frac{1}{3}}};\quad (x^2-1)(x+2).$$

9. Since first derivatives are themselves functions of the fundamental variable, their first derivatives may be found. These last are called the *second derivatives* of the original function. This operation may be extended until we have third, fourth, fifth derivatives, etc. Second derivatives are usually indicated by the symbol $f''(x)$, and third derivatives by $f'''(x)$.

EXAMPLES.

Find the second and third derivatives of the functions given at the close of the preceding article.

10. *To find the value of*

$$p_0x^n + p_1x^{n-1} + p_2x^{n-2} + \cdots + p_{n-1}x + p_n$$

after x *has been increased by the quantity* h.

Let $f(x) = p_0x^n + p_1x^{n-1} + \cdots + p_{n-1}x + p_n$;

then
$$f(x+h) = p_0(x+h)^n + p_1(x+h)^{n-1} + \cdots$$
$$+ p_{n-1}(x+h) + p_n.$$

Expanding each term and arranging the result according to the ascending powers of h, we have

$$\begin{aligned}&p_0x^n + p_1x^{n-1} + \cdots + p_{n-1}x + p_n\\ &\quad + h[np_0x^{n-1} + (n-1)p_1x^{n-2} + \cdots + 2p_{n-2}x + p_{n-1}]\\ &\quad + \frac{h^2}{\underline{|2}}[n(n-1)p_0x^{n-2} + (n-1)(n-2)p_1x^{n-3} + \cdots + 2p_{n-2}]\\ &\quad + \cdots\cdots\cdots\cdots\cdots\cdots\\ &\quad + \frac{h^n}{\underline{|n}}[n(n-1)\cdots(2)(1)p_0].\end{aligned}$$

The part of this expression which does not contain h is seen to be $f(x)$; the coefficient of the first power of h is the *first derivative* of $f(x)$; the coefficient of $\frac{h^2}{\underline{|2}}$ is the *second derivative* of $f(x)$, and so on.

Hence we have

$$f(x+h) = f(x) + h \cdot f'(x) + \frac{h^2}{\underline{|2}}f''(x) + \frac{h^3}{\underline{|3}}f'''(x) + \cdots.$$

Since n is positive and integral, it is evident that the series ends with the term which contains the nth derivative. This is a particular case of a theorem known as *Taylor's Theorem.*

11. We can now show the continuity of a rational integral function between any two finite values of x. By this we mean that the value of the function changes gradually as x changes gradually from one value to another.

From the preceding article

$$f(x+h)-f(x)=hf'(x)+\frac{h^2}{\underline{|2}}f''(x)+\cdots.$$

Since n is positive and integral, the second member of this equation consists of a finite number of terms; moreover, from the mode of formation of the coefficients $f'(x)$, $f''(x)$, etc., none of them can be infinite or imaginary for any real finite value of x; hence by supposing h to diminish, we can make the value of the second member as small as we please; hence, also, the first member may be made infinitely small, vanishing with h. The function is therefore continuous.

Again, since $\dfrac{f(x+h)-f(x)}{h}=f'(x)$ when h diminishes without limit, if h is positive, $f(x+h)-f(x)$ and $f'(x)$ must have the same sign. Consequently, when $f'(x)$ is positive, $f(x)$ is increasing with x; and when $f'(x)$ is negative, $f(x)$ diminishes as x increases.

12. *To find the remainder when any rational integral function of* x *is divided by* x − a.

Let $f(x)$ denote any rational integral function of x. Suppose the process of dividing $f(x)$ by $x-a$ to be continued until we have a remainder, r, which does not contain x. Let q be the quotient.

Then $$f(x)=(x-a)q+r.$$

Making x equal to a, $f(a)=r$.

Hence *the remainder is the given function evaluated for* a.

Cor. As a special case suppose $f(x)$ is exactly divisible by $x-a$; then $r=0$; hence $f(a)=0$. Conversely, *if* f(x) *vanishes when* x = a, f(x) *is exactly divisible by* x − a.

13. *If a rational integral function of the* nth *degree vanishes for more than* n *values of the variable, the coefficient of each power of the variable must be zero.*

Let $f(x)=p_0x^n+p_1x^{n-1}+p_2x^{n-2}+\cdots+p_{n-1}x+p_n$,

and suppose that $f(x)$ vanishes when x is made equal to each of the values $a_1, a_2, a_3, \cdots a_n$, no two of these quantities being equal. By the preceding article $f(x)$ is exactly divisible by $x-a_1$, and we have

$$f(x)=(x-a_1)(p_0x^{n-1}+\cdots),$$

the second factor being of the $(n-1)$th degree.

Again, since $f(x)$ is divisible by $x-a_2$, and since $x-a_1$ and $x-a_2$ are prime to each other, the factor $(p_0x^{n-1}+\cdots)$ must be divisible by $x-a_2$, and we have

$$f(x)=(x-a_1)(x-a_2)(p_0x^{n-2}+\cdots).$$

Continuing the factoring in this manner, we have finally

$$f(x)=(x-a_1)(x-a_2)\cdots(x-a_n)p_0,$$

and $f(x)$ is thus resolved into n linear binomial factors *with one other* factor p_0.

Suppose now that $f(x)$ vanishes for a quantity b, this quantity not being equal to any of the quantities $a_1, a_2, \cdots a_n$.

Then $f(b)=(b-a_1)(b-a_2)(b-a_3)\cdots(b-a_n)p_0=0$.

Since none of the factors $b-a_1$, $b-a_2$, $\cdots b-a_n$ can equal zero, we must conclude that p_0 equals zero. Continuing this process, it is shown that each one of the coefficients $p_0, p_1, \cdots p_n$ must equal zero.

14. *If two rational integral functions of the* nth *degree are equal for more than* n *values of the variable, they are equal for all values of the variable.*

Suppose

$$p_0x_n+p_1x^{n-1}+\cdots+p_{n-1}x+p_n$$
$$=p'_0x^n+p'_1x^{n-1}+\cdots+p'_{n-1}x+p'_n,$$

x having more than n values.

Transposing,

$$(p_0-p'_0)x^n+(p_1-p'_1)x^{n-1}+\cdots$$
$$+(p_{n-1}-p'_{n-1})x+p_n-p'_n=0,$$

and we have a function of the nth degree vanishing for more than n values of the variable. Therefore by the preceding article,

$$p_0-p'_0=0,\ p_1-p'_1=0,\ \cdots p_n-p'_n=0;$$
$$i.e.\quad p_0=p'_0,\qquad p_1=p'_1,\ \text{etc.}$$

Hence the two expressions

$$p_0x^n+p_1x^{n-1}+\cdots+p_{n-1}x+p_n,$$
$$p'_0x^n+p'_1x^{n-1}+\cdots+p'_{n-1}x+p'_n,$$

are identical, and therefore are equal for all values of the variable.

15. *If* $$p_0x^n + p_1x^{n-1} + \cdots + p_{n-1}x + p_n$$
$$= p'_0x^n + p'_1x^{n-1} + \cdots + p'_{n-1}x + p'_n$$

for all values of x, *the coefficients of the like powers of* x *are equal and the expressions are identical.*

Since the expressions are equal for all values of x, they are equal when x is zero; in that case $p_n = p'_n$. Dropping p_n and p'_n and dividing by x,

$$p_0x^{n-1} + p_1x^{n-2} + \cdots + p_{n-2}x + p_{n-1}$$
$$= p'_0x^{n-1} + p'_1x^{n-2} + \cdots + p'_{n-1}x + p'_{n-1}.$$

Again making x equal to zero, $p_{n-1} = p'_{n-1}$; and continuing the operation as before, the coefficients of the like powers of x are shown to be equal.

The proposition just proved is known as the *Principle of Undetermined Coefficients.* The principle is much used for developing a function into a series in ascending powers of the variable. In the preceding demonstration we notice that neither expression contains more than $n+1$ terms, a finite number; also, that we might have written

$$p_n + p_{n-1}x + \cdots + p_1x^{n-1} + p_0x^n$$
$$= p'_n + p'_{n-1}x + \cdots + p'_1x^{n-1} + p'_0x^n.$$

If, now, instead of a finite number of terms, we have the series

$$p_n + p_{n-1}x + p_{n-2}x^2 + \cdots = p'_n + p'_{n-1}x + p'_{n-2}x^2 + \cdots,$$

and this equality holds for all values of x, the demonstration would proceed as before, with the conclusion that the coefficients of the like powers of x are equal.

To illustrate the use of the principle, let it be required to develop $\frac{x}{1+x}$ into a series including the third power of x.

Let $$\frac{x}{1+x}=A+Bx+Cx^2+Dx^3+\cdots.$$

Here we assume that the given function $\frac{x}{1+x}$ can be developed into a series in ascending powers of x; our problem is to discover what values the quantities A, B, C, etc., must have in order that the function may equal such a series. Multiplying both members of the equality by $1+x$,

$$x=A+(A+B)x+(B+C)x^2+(C+D)x^3+\cdots.$$

The first member may be written

$$0+x+0x^2+0x^3+\cdots.$$

Equating the coefficients of the like powers of x,

$$\begin{aligned} A=0,\ (A+B)&=1, &\therefore\ B&=1;\\ B+C&=0, &\therefore\ C&=-1;\\ C+D&=0, &\therefore\ D&=1. \end{aligned}$$

The coefficients are now determined. Placing their values in the assumed series,

$$\frac{x}{1+x}=x-x^2+x^3-\cdots.$$

Ex. 2. Develop $\frac{1}{(1+x)^{\frac{1}{2}}}$.

Assume

$$\frac{1}{(1+x)^{\frac{1}{2}}}=A+Bx+Cx^2+Dx^3+Ex^4+\cdots.$$

Squaring both members of this equality,

$$\frac{1}{1+x} = \begin{array}{r|r|r|r|l} A^2 + AB & x + AC & x^2 + AD & x^3 + AE & x^4 + \cdots; \\ AB & B^2 & BC & BD & \\ & AC & BC & C^2 & \\ & & AD & BD & \\ & & & AE & \end{array}$$

the quantities at the left of any vertical line being the terms of the complete polynomial coefficient of the power of x written at the right of that line. Multiplying both members of the equality by $1+x$,

$$1 = \begin{array}{r|r|r|r|l} A^2 + AB & x + AC & x^2 + AD & x^3 + AE & x^4 + \cdots. \\ AB & B^2 & BC & BD & \\ A^2 & AC & BC & C^2 & \\ & 2AB & AD & BD & \\ & & B^2 + 2AC & AE & \\ & & & 2AD + 2BC & \end{array}$$

Equating the coefficients of the like powers of x, as in the first example,

$$A^2 = 1,$$
$$2AB + A^2 = 0,$$
$$2AC + 2AB + B^2 = 0,$$
$$2AD + 2BC + 2AC + B^2 = 0,$$
$$2AD + 2BC + 2BD + 2AE + C^2 = 0.$$

Thus we have a set of five simultaneous equations from which to determine the values of the five unknown quantities involved. If these values be substituted in the assumed series,

$$\frac{1}{(1+x)^{\frac{1}{2}}} = 1 - \tfrac{1}{2}x + \tfrac{3}{8}x^2 - \tfrac{5}{16}x^3 + \tfrac{35}{128}x^4 - \cdots.$$

Develop the following functions into series:

$$(3)\ \frac{a}{b+cx};\quad (4)\ \frac{1+x}{1+x+x^2};\quad (5)\ \frac{1-3x}{1+x^2}.$$

16. *To develop* $(a+x)^m$ *into a series,* m *being positive or negative, fractional or integral.*

Let $(a+x)^m = A + Bx + Cx^2 + Dx^3 + Ex^4 + \cdots,$

in which A, B, C, etc., are coefficients to be determined. Finding the first, second, and successive derivatives of the given function, and also of the assumed series, we have

$$\begin{aligned}
f'(x) &= m(a+x)^{m-1}\\
&= B + 2Cx + 3Dx^2 + 4Ex^3 + \cdots,\\
f''(x) &= m(m-1)(a+x)^{m-2}\\
&= 2C + 2\cdot 3Dx + 3\cdot 4Ex^2 + \cdots,\\
f'''(x) &= m(m-1)(m-2)(a+x)^{m-3}\\
&= 2\cdot 3D + 2\cdot 3\cdot 4Ex + \cdots,\\
f^{\text{iv}}(x) &= m(m-1)(m-2)(m-3)(a+x)^{m-4}\\
&= 2\cdot 3\cdot 4E + \cdots.
\end{aligned}$$

Since x is a variable, and these equations therefore true for all values of x, they are true when x equals zero; in that case we have

$$A = a^m;\quad B = ma^{m-1};\quad 2C = m(m-1)a^{m-2},$$

and $C = \dfrac{m(m-1)}{2}a^{m-2};\ 2\cdot 3D = m(m-1)(m-2)a^{m-3}$

and $D = \dfrac{m(m-1)(m-2)}{\underline{|3}}a^{m-3};$ and so on.

Substituting in the assumed series the coefficients as thus determined,

$$(a+x)^m = a^m + ma^{m-1}x + \frac{m(m-1)}{2}a^{m-2}x^2$$
$$+\frac{m(m-1)(m-2)}{\underline{3}}a^{m-3}x^3$$
$$+\frac{m(m-1)(m-2)(m-3)}{\underline{4}}a^{m-4}x^4+\cdots.$$

This is the *Binomial Theorem*, with a special case of which the student is already familiar. Comparison of several successive terms in this series enables us to see the law of the formation of each term so that we can write any specified term without first writing all the preceding ones. The nth term is seen to be

$$\frac{m(m-1)(m-2)\cdots(m-(n-2))}{\underline{|n-1}}a^{m-(n-1)}x^{n-1};$$

and the $(n+1)$th term is

$$\frac{m(m-1)(m-2)\cdots(m-(n-1))}{\underline{|n}}a^{m-n}x^n.$$

17. Suppose $f(x)=(1+x)^m$, in which m may be positive or negative, fractional or integral.

If this function be developed by the Binomial Theorem, we have

$$(1+x)^m = 1 + mx + \frac{m(m-1)}{2}x^2$$
$$+\frac{m(m-1)(m-2)}{\underline{|3}}x^3+\cdots$$

$$+\frac{m(m-1)\cdots(m-(n-2))}{\lfloor n-1}x^{n-1}$$

$$+\frac{m(m-1)\cdots(m-(n-1))}{\lfloor n}x^{n}+\cdots.$$

Now it is evident that no finite value of x can render $(1+x)^m$ infinite; hence no value of x is admissible which will render the infinite series

$$1+mx+\frac{m(m-1)}{2}x^2+\cdots$$

infinite in value.

An infinite series is said to be *convergent* when the sum of the first n terms cannot numerically exceed some finite quantity, however great n may be. If by taking n large enough the series can be made numerically greater than any finite quantity, the series is said to be *divergent.*

We have, then, to inquire whether any finite value of x renders the series

$$1+mx+\frac{m(m-1)}{2}x^2+\cdots$$

divergent.

First consider the geometrical series

$$1+x+x^2+x^3+\cdots.$$

The sum of n terms of this series is $\frac{1-x^n}{1-x}$. Let $x<1$; then as n increases, x^n diminishes, and by taking n sufficiently great we can make x^n as small as we please. Hence $\frac{1-x^n}{1-x}$ tends to the limit $\frac{1}{1-x}$ as n is increased.

Let $x>1$; then by taking n indefinitely large $\frac{1-x^n}{1-x}$, the sum of n terms becomes indefinitely large. The given series is therefore convergent for values of x less than 1, and divergent for values greater than 1.

Similarly, the sum of n terms of the series

$$1 - x + x^2 - x^3 + \cdots$$

is

$$\frac{1-(-1)^n x^n}{1+x}.$$

When $x < 1$, the limit of the sum of this series is $\frac{1}{1+x}$, and the series is convergent as in the first case; but when $x > 1$, x^n increases as n increases, and the sum of the series has no limit; *i.e.* the series is divergent.

18. The following proposition can now be established:

If, in a series of positive terms, as

$$\mu_1 + \mu_2 + \mu_3 + \cdots + \mu_n + \mu_{n+1} + \cdots,$$

the ratio $\frac{\mu_{n+1}}{\mu_n}$ *be less than a certain quantity itself less than unity, for all values of* n *beyond a certain number, the series is convergent.*

Suppose k to be a fraction less than unity and greater than the greatest of the ratios

$$\frac{\mu_{n+1}}{\mu_n} \cdots \text{ beyond the number } n,$$

then

$$\frac{\mu_{n+1}}{\mu_n} < k, \qquad \therefore \; \mu_{n+1} < k\mu_n; \qquad (1)$$

$$\frac{\mu_{n+2}}{\mu_{n+1}} < k, \qquad \therefore \; \mu_{n+2} < k\mu_{n+1}; \qquad (2)$$

$$\frac{\mu_{n+3}}{\mu_{n+2}} < k, \qquad \therefore \; \mu_{n+3} < k\mu_{n+2}; \qquad (3)$$

and so on.

Since $k\mu_n > \mu_{n+1}$, if we substitute $k\mu_n$ for μ_{n+1} in (2), we have $\mu_{n+2} < k^2\mu_n$. Making a similar substitution in (3), etc., and adding the corresponding terms of these inequalities,

$$(\mu_{n+1} + \mu_{n+2} + \mu_{n+3} + \cdots) < (k\mu_n + k^2\mu_n + k^3\mu_n + \cdots).$$

But since k is a proper fraction, $k + k^2 + k^3 + \cdots$ is a converging series with $\frac{k}{1-k}$ for its limit. The other factor μ_n decreases as n increases, tending to zero as its limit. Hence the second member of the above inequality has zero as its limit. The first member of the inequality is the sum of the series after the nth term, and since it is less than $\frac{k\mu_n}{1-k}$, it must also have zero as its limit. Now the sum of the n terms preceding $\mu_{n+1} + \mu_{n+2} + \cdots$ is a finite quantity, being the sum of a *finite* number of *finite* terms. Adding this to the series beginning with μ_{n+1}, the limit of the entire series is seen to be a finite quantity; hence the conclusion that the series is convergent.

On the other hand, if the ratio $\frac{\mu_{n+1}}{\mu_n}$ is greater than 1 for all values of n beyond a certain number, the series is divergent.

Proceeding as before, let $k > 1$; then

$$(\mu_{n+1} + \mu_{n+2} + \cdots) > \mu_n(k + k^2 + k^3 + \cdots).$$

Now both factors in the second member of this inequality increase without limit; consequently the first member, which is always greater than the second member, must increase without limit; the series is therefore divergent.

In the preceding argument we have assumed that the terms μ_1, μ_2, μ_3, etc., are all positive. The conclusions reached may be shown to hold in case the terms of the series are alternately positive and negative; for k now becomes negative, and the series will be convergent or divergent according as $-k$ is $<$ or >1.

EXAMPLES.

Determine whether the following series are convergent or divergent:

1. $\dfrac{x}{1\cdot 2}+\dfrac{x^2}{2\cdot 3}+\dfrac{x^3}{3\cdot 4}+\dfrac{x^4}{4\cdot 5}+\cdots.$

2. $1+\dfrac{2^2}{\underline{|2}}+\dfrac{3^2}{\underline{|3}}+\dfrac{4^2}{\underline{|4}}+\cdots.$

3. $\sqrt{\tfrac{1}{2}}+\sqrt{\tfrac{2}{3}}+\sqrt{\tfrac{3}{4}}+\sqrt{\tfrac{4}{5}}+\cdots.$

4. $1+3x+5x^2+7x^3+9x^4+\cdots.$

19. Returning to the infinite series which $(1+x)^m$ becomes when developed, the ratio $\frac{\mu_{n+1}}{\mu_n}$ is seen to be $\dfrac{m-n+1}{n}x$, which equals $\left(-1+\dfrac{m}{n}+\dfrac{1}{n}\right)x$. The first factor tends to the limit -1 as n increases indefinitely; if $x<1$, the limit of the entire expression is therefore some proper fraction, and the series is convergent; but if $x>1$, the series is divergent. Hence values of x greater than 1 are not admissible in the development of $(1+x)^m$.

The more general function $(a+x)^m$ may be written $a^m\left(1+\dfrac{x}{a}\right)^m$. Let $\dfrac{x}{a}=z$; applying the above results to

$(1+z)^m$, z must be less than 1; otherwise the series would be divergent. But if z is less than 1, x is less than a; hence the Binomial Theorem does not hold when $x > a$.

EXAMPLES.

Develop into series in x:

$$(a-x)^{-2}; \qquad \frac{1}{b+x}; \qquad (a+2x)^{-4}; \qquad (2+x^2)^{\frac{1}{3}}.$$

II. THEORY OF EQUATIONS.

20. *An Equation* is a statement of equality between two functions of one or more quantities. Thus

$$f(a, x, b, y, \cdots) = \phi(a, x, b, y, \cdots),$$

or

$$F(a, x, b, y, \cdots) = 0,$$

represents all statements made by means of mathematical symbols and called equations. If the functions are regarded (1) as functions of one quantity only, and if (2) they are differently constituted, each value of the quantity for which the statement is true is called a *root* of the equation. If condition (1) holds and (2) does not, the equation reduces to an identity; *e.g.* $x-5=x-5$ is an identity, and the statement is true for *all* values of x. If condition (2) holds and (1) does not, the equation is said to be indeterminate, and the statement is true for an indefinite number of sets of values of the quantities involved; *e.g.* the equation $2x-3y+1=0$ is indeterminate if viewed as an equation in x and y. We may assign to x all values that we please, and by properly assigning values to y, may keep the statement of equality true. Thus, if $x=1$, y must equal 1; if $x=2$, $y=\frac{5}{3}$; if $x=-1$, $y=-\frac{1}{3}$, etc. In what follows we shall consider equations in which the above stated conditions (1) and (2) hold, together with a third condition that the functions shall be rational integral algebraic functions; and our problem is: to discover the roots of such equations,—their number and character and values.

Except when otherwise defined, the symbol $f(x)$ will be used in the following articles to denote the rational integral function

$$x^n + p_1x^{n-1} + p_2x^{n-2} + \cdots + p_{n-1}x + p^n,$$

in which the coefficients p_1, p_2, etc., are the ratios obtained by dividing the original coefficients by the coefficient of x^n.

21. *Assuming that every equation of the form* $f(x) = 0$ *has a root, real or imaginary, if the equation be of the* n*th degree, it has* n *roots and no more.*

Admitting that some of the coefficients of $f(x)$ are not zero, it follows from Art. 13 that $f(x)$ cannot vanish for more than n values of x; *i.e.* $f(x) = 0$ cannot have more than n roots. Now let α_1 be a root, then $f(\alpha_1) = 0$; also, by a preceding article, $f(x)$ is exactly divisible by $x - \alpha_1$, and we have

$$f(x) = (x - \alpha_1)\, f_1(x) = 0,$$

in which the factor $f_1(x)$ is of the $(n-1)$th degree.

Dividing both members of this equation by $x - \alpha_1$, we have $f_1(x) = 0$.

Suppose α_2 is a root of $f_1(x) = 0$; then as before,

$$f_1(x) = (x - \alpha_2)\, f_2(x) = 0;$$

in which $f_2(x)$ is of the $(n-2)$th degree.

Repeating this operation, we have finally an equation of the first degree, and

$$f(x) = (x - \alpha_1)(x - \alpha_2)(x - \alpha_3) \cdots (x - \alpha_n) = 0.$$

Thus the first member of the equation is resolvable into n linear binomial factors, and the second term of each binomial is a root of the equation.

Cor. 1. *Two or more of the roots may be equal: the binomial factors remain* n *in number, and we still speak of the equation as having* n *roots.*

Cor. 2. *If the multiplication indicated by*

$$(\mathrm{x} - \alpha_1)(\mathrm{x} - \alpha_2) \cdots (\mathrm{x} - \alpha_{\mathrm{n}-1})(\mathrm{x} - \alpha_{\mathrm{n}})$$

be performed, it is evident that the term $\mathrm{p_n}$ *of the original polynomial equals*

$$(-1)^{\mathrm{n}} \alpha_1 \alpha_2 \alpha_3 \cdots \alpha_{\mathrm{n}-1} \alpha_{\mathrm{n}}.$$

Hence $\mathrm{p_n}$, *the absolute term in the equation* $\mathrm{f(x)} = 0$, *is exactly divisible by each root of the equation.*

22. *If the coefficients of the equation are all real, imaginary roots enter it in pairs;* i.e. *if there be a root of the form* $\alpha + \beta\sqrt{-1}$, *there must be an accompanying root of the form* $\alpha - \beta\sqrt{-1}$.

Let $\alpha + \beta\sqrt{-1}$ be a root of $f(x) = 0$. Then $\alpha - \beta\sqrt{-1}$ is also a root if $f(x)$ is exactly divisible by

$$\{x - (\alpha + \beta\sqrt{-1})\}\{x - (\alpha - \beta\sqrt{-1})\},$$

which equals $(x - \alpha)^2 + \beta^2$. First suppose that $f(x)$ is not exactly divisible by $(x - \alpha)^2 + \beta^2$; let q be the quotient, and $rx + r'$ the remainder.

Then $f(x) = \{(x-\alpha)^2 + \beta^2\}q + rx + r'$.

Since $\alpha + \beta\sqrt{-1}$ is a root,

$$f(\alpha + \beta\sqrt{-1}) = 0.$$

Substituting this value of x in the second member also,

$$(\alpha + \beta\sqrt{-1} - \alpha)^2 + \beta^2 = 0;$$

hence $r(\alpha + \beta\sqrt{-1}) + r' = 0$.

To render this last statement true the imaginary and real parts must be separately equated to zero, and we have $r\alpha + r' = 0$ and $r\beta\sqrt{-1} = 0$. Hence $r = 0$ and $r' = 0$; hence $f(x)$ is exactly divisible by $(x-\alpha)^2 + \beta^2$. $\alpha - \beta\sqrt{-1}$ is therefore a root of $f(x) = 0$.

COR. 1. *It follows from the above theorem that an equation of an odd degree must have at least one real root; but an equation of an even degree may have all of its roots imaginary.*

COR. 2. *If the coefficients of an equation are rational, surd roots must enter it in pairs.* That is, if there be a root of the form $\alpha + \sqrt{\beta}$, there must be an accompanying root of the form $\alpha - \sqrt{\beta}$. The proof is similar to that given in the case of imaginaries.

EXAMPLES.

Form the equations whose roots are:

$$-1, 2, 5; \ -\tfrac{2}{3}, \tfrac{3}{2}, 4; \ a, -3, 3.$$

Form without surds the equations whose roots are:

$$\sqrt{3}, 1+\sqrt{3}, 3; \ \sqrt{2}-2, -\sqrt{4}, -\sqrt{4}.$$

Form with real coefficients the equations whose roots are:

$$4\sqrt{-1},\ 1,\ -1;\quad 2-\sqrt{-3},\ \tfrac{1}{2},\ -2.$$

Solve the equation

$$x^3-8x^2+6x+52=0,$$

one of the roots being $5-\sqrt{-1}$.

Solve the equation

$$x^5-4x^3+4x^2+4x-8=0,$$

one of the roots being $\sqrt{2}$ and another $1+\sqrt{-1}$.

What can be affirmed of the roots of equations under the following conditions:

1. Coefficients all positive.
2. Coefficients of the even powers of x preceded by the same sign, and the coefficients of the odd powers preceded by the contrary sign.
3. Equation containing only the even powers of x, and the coefficients having the same sign.
4. Equation containing only the odd powers of x, and the coefficients having the same sign.

NOTE. The absolute term is regarded as an even power term.

23. In any series of quantities given with their signs a succession of two like signs is called a ***permanence*** of signs, and a succession of two unlike signs, a ***variation***. Thus if we write in order the signs of the terms of the polynomial x^3-x^2-1, we have $+\,-\,-$; the first and second constitute a variation, the second and third a permanence. The following is Descartes' ***Rule of Signs*** for positive roots:

No equation can have more positive roots than it has variations in the terms of its first member.

Suppose that the signs of the terms of the polynomial $f(x)$ are

$$+ - + - - - + + -.$$

The signs of a binomial of the form $x - \alpha$, where α is a positive root, are $+\ -$. We proceed to show that if the polynomial be multiplied by the binomial, there will be at least one more variation in the product than in the original polynomial. Writing down the signs only of the terms in the multiplication, and placing an ambiguous sign wherever two terms with different signs are to be added, we have

$$\begin{array}{cccccccccc}
+ & - & + & - & - & - & + & + & - & \\
+ & - & & & & & & & & \\
\hline
+ & - & + & - & - & - & + & + & - & \\
 & - & + & - & + & + & + & - & - & + \\
\hline
+ & - & + & - & \pm & \pm & + & \pm & - & +
\end{array}$$

In this product, we notice that

(1) an ambiguity occurs whenever + follows +, or − follows −, in the original polynomial;

(2) the signs before and after an ambiguity, or set of ambiguities, are unlike;

(3) a change of sign is introduced at the end;

(4) in general, the sum of the permanences and variations is $n+1$ if the sum was n in the original polynomial.

On account of (2), the number of variations is not diminished, and in whatever way this series of signs is read, the number of variations is at least one greater than in the original series; and this is true even if we omit the ambiguous signs altogether in reading, the variation *being gained* at the end. If the original series ends with a permanence, it will be seen, as before, that there is an additional variation. Thus we may have

$$\begin{array}{ccccccccc}
+ & - & + & - & - & - & + & + & \\
+ & - & & & & & & & \\
\hline
+ & - & + & - & - & - & + & + & \\
 & - & + & - & + & + & + & - & - \\
\hline
+ & - & + & - & \pm & \pm & + & \pm & -
\end{array}$$

Therefore, if we suppose the binomial factors formed from the negative and imaginary roots to have been already multiplied together so that $\phi(x)$ contains only these roots, each factor of the form $x-\alpha$ corresponding to a positive root introduces at least one variation.

24. If in the identical equation

$$f(x)=(x-\alpha_1)(x-\alpha_2)\cdots(x-\alpha_n),$$

we write $-x$ for x, we shall have

$$\begin{aligned}f(-x)&=(-x-\alpha_1)(-x-\alpha_2)\cdots(-x-\alpha_n)\\ &=(-1)^n(x+\alpha_1)(x+\alpha_2)(x+\alpha_3)\cdots(x+\alpha_n).\end{aligned}$$

Whence it appears that $-\alpha_1, -\alpha_2, \cdots -\alpha_n$ are the roots of $f(-x)=0$, and therefore the roots of $f(-x)=0$ are equal to those of $f(x)=0$ with the signs changed; *i.e.* the positive roots of $f(-x)=0$ are the negative roots of $f(x)=0$. But by the first rule, $f(-x)=0$ cannot have more positive roots than there are variations in the signs of the terms of $f(-x)$; which is the same as saying that $\mathrm{f(x)}=0$ *cannot have more negative roots than there are variations in the signs of* $\mathrm{f(-x)}$. This is Descartes' Rule of Signs for negative roots.

The student is cautioned against drawing the conclusion that an equation will necessarily have as many positive roots as there are variations in the signs of $f(x)$, and as many negative roots as there are variations in the signs of $f(-x)$. It will be observed that, in general. Descartes' Rule merely furnishes us with *outside limits* of the number of positive and negative roots; other considerations must determine the number of roots of each kind as well as their value.

EXAMPLES.

1. Find the superior limit to the number of real roots of the equation

$$x^3 + 3x + a = 0,$$

(1) when a is positive, and (2) when a is negative.

2. Find the superior limit to the number of real roots of the equation

$$x^8 - ax^5 + bx - c = 0,$$

a, b, and c being essentially positive.

[The equations of examples 1 and 2 are of great importance in Astronomy.]

3. Show that the equation

$$x^7 - x^4 + x^3 - 1 = 0$$

has at least four imaginary roots.

4. Find the least possible number of imaginary roots of the equation

$$x^9 - x^5 + x^4 + x^2 + 2 = 0.$$

5. Examine the equation

$$x^n - 1 = 0$$

for real and imaginary roots (1) when n is even, and (2) when n is odd.

6. Examine $$x^n+1=0,$$
(1) when n is even, and (2) when n is odd.

7. Given $$x^3 \pm ax+b=0,$$
where a and b are essentially positive; show that in the first equation we have one negative and two imaginary roots; and in the second, one negative root, while the other two are both imaginary or both positive.

8. Apply Descartes' Rule of Signs to the equations described in 1, 2, 3, 4, Art. 22.

25. Whenever it is required to find the value of $f(x)$ for some particular value of x, the result may be obtained more readily than by direct substitution. Thus if

$$x^4+p_1x^3+p_2x^2+p_3x+p_4$$

is to be evaluated for $x=\alpha$, the process is as follows: write the detached coefficients with their signs, multiply 1, the coefficient of the highest power of x, by α, and add the result to the next coefficient; multiply this sum by α, add the product to the third coefficient, and so proceed till the last term has been used. The last sum is the value of $f(x)$. The reasons for this rule will be found in considering the process itself as exhibited below.

$$\begin{array}{rrrr} 1 & +p_1 & +p_2 & +p_3 \\ & \alpha & \alpha^2+p_1\alpha & \alpha^3+p_1\alpha^2+p_2\alpha \\ \cline{2-4} & \alpha+p_1 & \alpha^2+p_1\alpha+p_2 & \alpha^3+p_1\alpha^2+p_2\alpha+p_3 \end{array}$$

$$\begin{array}{r} +p_4 \\ \alpha^4+p_1\alpha^3+p_2\alpha^2+p_3\alpha \\ \hline \alpha^4+p_1\alpha^3+p_2\alpha^2+p_3\alpha+p_4 \end{array}$$

It is evident, also, that if any given function of x is incomplete, *i.e.* if any power of x is absent, its coefficient 0 must be written in the series of coefficients.

Ex. 1. Find the value of $x^5 - 2x^3 + 3x + 4$ when $x = 2$.

$$\begin{array}{rrrrrrl}
1 & 0 & -2 & 0 & +3 & +4 & (2 \\
 & 2 & 4 & 4 & 8 & 22 & \\
\hline
 & 2 & 2 & 4 & 11 & 26 &
\end{array}$$

$$\therefore f(2) = 26.$$

Ex. 2. Determine whether -3 is a root of

$$x^4 + 7x^3 + 5x^2 - 31x - 30 = 0.$$

$$\begin{array}{rrrrrl}
1 & +7 & +5 & -31 & -30 & (-3 \\
 & -3 & -12 & 21 & +30 & \\
\hline
 & 4 & -7 & -10 & 0 &
\end{array}$$

Since $f(-3) = 0$, -3 is a root of $f(x) = 0$.

26. The process of the preceding article also enables us to divide $f(x)$ by $x - \alpha$ when α is a root of $f(x) = 0$. Let $f(x)$ be evaluated for α as before; the required quotient $f(x) \div (x - \alpha)$ will be, supposing $f(x)$ is of the fourth degree,

$$x^3 + (\alpha + p_1)x^2 + (\alpha^2 + p_1\alpha + p_2)x + (\alpha^3 + p_1\alpha^2 + p_2\alpha + p_3).$$

The function is one degree lower than $f(x)$; and the coefficients of x after the first are the successive sums obtained in the process of evaluating, the last sum vanishing, since α is now supposed to be a root of $f(x) = 0$. That this is the quotient when $f(x)$ is divided by $x - \alpha$ appears at once by multiplying this new function by $x - \alpha$; *thus:*

$$\begin{array}{r}
x^3+(\alpha+p_1)x^2+(\alpha^2+p_1\alpha+p_2)x+(\alpha^3+p_1\alpha^2+p_2\alpha+p_3) \\
x-\alpha \\
\hline
x^4+(\alpha+p_1)x^3+(\alpha^2+p_1\alpha+p_2)x^2+(\alpha^3+p_1\alpha^2+p_2\alpha+p_3)x \\
-\alpha x^3-(\alpha^2+p_1\alpha)x^2-(\alpha^3+p_1\alpha^2+p_2\alpha)-(\alpha^4+p_1\alpha^3+p_2\alpha^2+p_3\alpha) \\
\hline
x^4+p_1x^3+p_2x^2+p_3x-(\alpha^4+p_1\alpha^3+p_2\alpha^2+p_3\alpha).
\end{array}$$

Since $\alpha^4+p_1\alpha^3+p_2\alpha^2+p_3\alpha+p_4=0,$

or $\alpha^4+p_1\alpha^3+p_2\alpha^2+p_3\alpha=-p_4,$

we have finally for the product

$$x^4+p_1x^3+p_2x^2+p_3x+p_4,$$

the original function.

To illustrate the use of this principle let it be required to obtain an equation of lower degree than

$$x^4+7x^3+5x^2-31x-30=0.$$

Evaluating for -3 as above, we may write at once,

$$x^3+4x^2-7x-10,$$

the quotient that would be obtained by dividing

$$x^4+7x^3+5x^2-31x-30$$

by $x+3$ in the usual way.

27. A root of $f(x)=0$ has already been defined as a value of x which will render the statement, $f(x)=0$, true; *i.e.* if α be a root, $f(\alpha)$ vanishes. If any other value not a root, as β, be assigned to x, the function thus evaluated will not vanish; *i.e.* we shall have $f(\beta)$ equal to some quantity not zero. Now if x be viewed as a variable and be conceived to pass through all real values from $-\infty$ to $+\infty$, it is evident (1) that we shall have as many values for the function as for the variable,

(2) that in general the value of the function will not be zero, and (3) that whenever x reaches a *root* value the function will be zero. It is possible to represent graphically the changes in the function corresponding to the changes in the variable. Draw a horizontal line with a second straight line at right angles to the first. Call the horizontal line, XX', or the *X-axis;* the vertical line YY', or the *Y-axis;* and their point of intersection O, or the origin. Beginning at O as the zero point, lay off with any convenient unit of length the positive values of x to the right on the X-axis; and the negative values to the left on this axis. At the end, remote from O, of this line which represents the value of x, draw a perpendicular (using the same unit of length) to represent the corresponding value of $f(x)$. The perpendicular is to be drawn upward from the X-axis in case $f(x)$ is positive, and downward from this axis when $f(x)$ is negative. Thus a point is located in the plane, and if many values be given to x, — any two consecutive values differing but little from each other, — we shall have a correspondingly large number of points with small distances separating them; and the assemblage of points will ultimately form a continuous line, straight or curved, which may be called the *Graph* of $f(x)$. The values of x are called *abscissas,* and the values of $f(x)$, *ordinates;* the two together are known as *co-ordinates.*

To illustrate: let us take the equation $x+2=0$ and construct the graph of $f(x)$, having in this case $f(x)=x+2$. When $x=0$, $f(x)=+2$; when $x=+1$, $f(x)=+3$; when $x=+2$, $f(x)=+4$; etc. Taking the first pair of values, since $x=0$ we have no distance to measure off on the X-axis, and since $f(x)=+2$, we *measure upward 2* units, thus locating the graphic point P_1. Again, measuring 1 unit to the right and 3 units upward, we have the point P_2. If $x = -1$, $f(x) = +1$; hence, measuring 1 unit to the left on the X-axis and then 1 unit upward, we have a third point P_3.

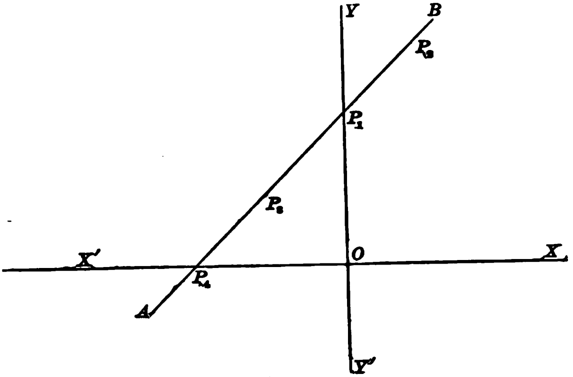

FIG. 1.

If many points are located in this way, they will be found to lie on the straight line determined by any two of them. (It will be shown in Analytic Geometry that the graph is always a straight line when $f(x)$ is of the first degree.) We shall therefore draw the straight line AB passing through the points determined. For the purposes of these Lessons especial attention must be given to the fact that *the distance from the origin O to the point where the graph cuts the X-axis represents a root of* $f(x) = 0$. This is evidently true, since only root values of x make $f(x)$ vanish, and when $f(x)$ vanishes, we have to measure neither up nor down from the X-axis; that is, the points determined are the particular points of the graph which lie on the X-axis. In the above example the distance $OP_4 (= -2)$ represents the one root of the equation $x + 2 = 0$.

It will now be seen that having any equation $f(x)=0$, of any degree, if we can construct its graph, the several distances from O to the points where the graph cuts the X-axis will be the real roots of the equation; we are thus furnished with one method of solution of equations.

Ex. 2. Construct the graph of $f(x)$ when

$$f(x)=x^2+x-6;$$

and by means of it show that the roots of the equation $x^2+x-6=0$ are -3 and $+2$.

Writing a value of x and the accompanying value of $f(x)$ in parenthesis, we have, for instance, the following pairs of values: $(0, -6)$; $(1, -4)$; $(-1, -6)$; $(-3, 0)$; $(2, 0)$. Locating the corresponding points as in the first example, and drawing a curve through them, the graph is as seen in Fig. 2. If many points be located, they will be found to lie on the curve as drawn; there are no small bends or other irregularities in the curve. Moreover, the graph of any particular quadratic function, when determined in this experimental manner, will be found to be similar to the graph in Fig. 2. In Analytic Geometry it is proven deductively that the graphs of *all* functions of the form ax^2+bx+c *are* similar to that of Fig. 2.

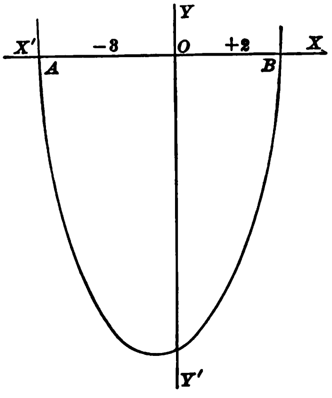

FIG. 2.

28. If $\frac{25}{4}$ be added to the absolute term of the equation in the above example, we have $x^2 + x + \frac{1}{4} = 0$, an equation whose roots are $-\frac{1}{2}$ and $-\frac{1}{2}$. The effect upon the graph is a change in its position with respect to the X-axis. This axis has moved parallel to its first position until it just touches the lowest point of the curve, the two unequal intercepts, -3 and $+2$, on the X-axis being replaced by the two coincident intercepts $-\frac{1}{2}$ and $-\frac{1}{2}$. If a still greater change be made in the absolute term of $x^2 + x - 6$, the graph will no longer touch the X-axis; the two equal roots of the equation will be replaced by two imaginary roots, and the graph is cut by the X-axis in two imaginary points. For instance, if the absolute term $\frac{1}{4}$ were changed to $\frac{1}{3}$, we should have the equation $x^2 + x + \frac{1}{3} = 0$, whose roots are $-\frac{1}{2} \pm \frac{1}{2}\sqrt{-\frac{1}{3}}$.

Ex. 3. Construct the graph of $x^2 - 4x + 4$, and show that the equation

$$x^2 - 4x + 4 = 0$$

has two equal roots.

Ex. 4. Construct the graph of $x^2 - 2x + 4$, and show that the equation

$$x^2 - 2x + 4 = 0$$

has no real roots.

Compare examples 3 and 4, and notice that a change in the position of the graph with respect to the X-axis may be accomplished in other ways besides a change in the absolute term.

Ex. 5. Construct the graph of $x^3 - 6x^2 + 11x - 6$, and by means of it show the roots of the equation

$$x^3 - 6x^2 + 11x - 6 = 0.$$

Ex. 6. Construct the graph of x^3+2x^2+4x+8, and show that the equation

$$x^3+2x^2+4x+8=0$$

has only one real root, and that this root is negative.

Ex. 7. Construct the graph of

$$x^4-6x^3+13x^2-12x+4,$$

and show that the equation formed by equating this function to zero has two pairs of equal roots.

Ex. 8. Form an equation of the fourth degree in real coefficients that shall have two of its roots zero and a third root $\frac{1}{2}\sqrt{-1}$; and exhibit the roots graphically.

29. f(x), *of the equation* f(x) = 0, *will change sign when* x *passes through any real root, as* α, *if there is an odd number of roots having the value* α; *but if there is an even number of such roots,* f(x) *will not change sign.*

To make the case perfectly general, let $\alpha_1, \alpha_2 \cdots \alpha_{n-2m}$ be the real roots, and let $\beta_1, \beta_2 \cdots \beta_{2m}$ be the imaginary roots, and suppose

$$\phi(x)=(x-\beta_1)(x-\beta_2)\cdots(x-\beta_{2m}).$$

Then we have

$$f(x)=(x-\alpha_1)(x-\alpha_2)\cdots(x-\alpha_{n-2m})\{\phi(x)\}=0.$$

As $f(x)$ is supposed to contain only real coefficients, $\phi(x)$ must be of even degree. No real value of x can cause $\phi(x)$ to change sign; for if we form the equation $\phi(x)=0$, it has only imaginary roots; therefore the graph of $\phi(x)$ does not cut the X-axis, and all the perpendiculars representing values of $\phi(x)$ are on the same side of the axis. Hence, in examining $f(x)$ for changes of sign, we need consider only the product

$$(x-\alpha_1)(x-\alpha_2)\cdots(x-\alpha_{n-2m}).$$

Suppose x to start with some value less than the least root, and to increase continuously until it becomes greater than the greatest root. So long as x is less than the least root, all the factors $(x-\alpha_1)$, $(x-\alpha_2)$, etc., are negative; but when x passes the value of the least root, say α_1, the factor containing that root becomes positive, and if there is no other root equal to α_1, this factor will be the only one which will change sign; consequently the sign of the entire product is changed. Further, if there is an odd number of roots having the value α_1, an *odd* number of factors will change sign, and the sign of the entire product is changed; but if there is an *even* number of roots having the value α_1, an even number of factors will change sign, and therefore the sign of the entire product remains unchanged. As x goes on increasing after it has passed the first root, it is evident that the factor containing this root cannot become negative again, and does not need to be considered in determining the subsequent signs of $f(x)$. The above argument is repeated in the case of each root $\alpha_2, \alpha_3, \cdots \alpha_{n-2m}$, and thus the theorem is established.

30. *If* f(x) *be evaluated in succession for two quantities, as* a *and* b, *and if the results,* f(a) *and* f(b), *have unlike signs, then the equation* f(x) = 0 *must have at least one real root between* a *and* b.

For $x=a$ and $x=b$, we have on the graph of $f(x)$ two points whose co-ordinates are $[a, f(a)]$ and $[b, f(b)]$, and since it is assumed that $f(a)$ and $f(b)$ have unlike signs, the two points must lie on opposite sides of the

X-axis. Now $f(x)$ is of the form $x^n + p_1x^{n-1} + \cdots + p_{n-1}x + p_n$, a rational integral algebraic function, and no finite value of x can render the function equal to ∞. Moreover, as shown in Art. 5, $f(x)$ varies continuously from $f(a)$ to $f(b)$; *i.e.* it passes through all intermediate values while x changes from a to b; hence the graph must cross the X-axis in order to connect the points $[a, f(a)]$ and $[b, f(b)]$, but at the point of crossing $f(x) = 0$, and there is therefore a root of $f(x) = 0$ between a and b.

Cor. The graph, by crossing the X-axis an *odd* number of times, may connect points lying on opposite sides of the axis. Therefore if $f(a)$ and $f(b)$ have unlike signs, there may be an odd number of roots of $f(x) = 0$ between the values a and b.

31. *If there exists no real quantity which, substituted for* x, *makes* f(x) *vanish, then* f(x) *must be positive for every real value of* x.

Since $f(x) = 0$ has no real roots, the graph of $f(x)$ must lie altogether on one side of the X-axis; and if we can determine the sign of $f(x)$ for one value of x, we shall know its sign for all values of x. The equation $f(x) = 0$, where

$$f(x) = x^n + p_1x^{n-1} + \cdots + p_{n-1}x + p_n,$$

may be written

$$x^n\left\{1 + p_1\frac{1}{x} + p_2\frac{1}{x^2} + \cdots + p_{n-1}\frac{1}{x^{n-1}} + p_n\frac{1}{x^n}\right\} = 0.$$

As x increases, the value of the function tends to become x^n, and when $x = +\infty$, $f(x) = (+\infty)^n$, a positive quantity infinitely great,

32. *Every equation of an odd degree has at least one real root of a sign opposite to that of its last term.*

Given

$$f(x) = x^n + p_1 x^{n-1} + \cdots + p_{n-1} x + p_n = 0,$$

n being an odd number. We have already seen that imaginary roots enter the equation in pairs; hence if $2m$ be the number of imaginary roots, the number of real roots is $n - 2m$, an odd number. It remains to show that one of these $n - 2m$ real roots is of opposite sign to that of p_n.

$$\text{If} \begin{cases} x = -\infty, & f(x) \text{ is negative;} \\ x = 0, & \text{the sign of } f(x) \text{ is the same as that of } p_n; \\ x = +\infty, & f(x) \text{ is positive.} \end{cases}$$

If p_n is positive, the graph of $f(x)$ must cross the X-axis between $-\infty$ and 0; *i.e.* there is one negative root; and if p_n is negative, the graph crosses the axis between 0 and $+\infty$; *i.e.* there is a root between 0 and $+\infty$. Thus the theorem is proved.

33. *Every equation of an even degree whose last term is negative has at least two real roots, one positive and one negative.*

Substituting as in the preceding demonstration, we have:

x	$f(x)$
$-\infty$	$+$
0	$-$
$+\infty$	$+$

Since $f(x)$ has changed sign between $-\infty$ and 0, and again between 0 and $+\infty$, there must exist at least one real negative root and one real positive root.

34. *If* $f(x)=0$ *has no equal roots,* $f(x)$ *and* $f'(x)$ *have no highest common divisor.*

Let $f(x)=(x-\alpha_1)(x-\alpha_2)\cdots(x-\alpha_n)$,

in which no two of the quantities $\alpha_1, \alpha_2, \ldots \alpha_n$ are equal to each other.

Then
$$\begin{aligned}f'(x)=&+(x-\alpha_2)(x-\alpha_3)\cdots(x-\alpha_n)\\&+(x-\alpha_1)(x-\alpha_3)\cdots(x-\alpha_n)+\cdots\\&+(x-\alpha_1)(x-\alpha_2)\cdots(x-\alpha_{n-1}).\end{aligned}$$

No one of the linear binomial factors of $f(x)$ is a factor of $f'(x)$; and since these linear factors are the prime factors, the two functions $f(x)$ and $f'(x)$ have no highest common divisor.

35. *If* $f(x)=0$ *has equal roots, the highest common divisor of* $f(x)$ *and* $f'(x)$, *when equated to zero, constitutes an equation which has for its roots these equal roots, and no others.*

Let α_1 be one of the m equal roots of $f(x)=0$, and let the other roots be $\alpha_2, \alpha_3, \ldots \alpha_{n-m+1}$; then

$$\begin{aligned}f(x)&=(x-\alpha_1)^m(x-\alpha_2)\cdots(x-\alpha_{n-m+1})\\&=(x-\alpha_1)^m\phi(x).\end{aligned}$$

Hence $f'(x)=m(x-\alpha_1)^{m-1}\phi(x)+(x-\alpha_1)^m\phi'(x)$.

By the preceding article, $\phi(x)$ and $\phi'(x)$ have no highest common divisor; therefore $(x-\alpha_1)^{m-1}$ is the highest common divisor of $f(x)$ and $f'(x)$. Putting it equal to zero, we have

$$(x-\alpha_1)^{m-1}=0,$$

an equation of the $(m-1)$th degree, having for its roots α_1 taken $m-1$ times. Thus the number of equal roots in the given equation is greater by one than the number obtained from the equation $(x-\alpha_1)^{m-1}=0$.

COR. 1. If $f(x)=0$ has two sets of equal roots, so that

$$f(x)=(x-\alpha_1)^m(x-\alpha_2)^q\phi(x),$$

the above process may be repeated, and the highest common divisor of $f(x)$ and $f'(x)$ will be found to be

$$(x-\alpha_1)^{m-1}(x-\alpha_2)^{q-1}.$$

The solution of the equation,

$$(x-\alpha_1)^{m-1}(x-\alpha_2)^{q-1}=0,$$

will evidently give roots having the values α_1 and α_2, $m-1$ roots with the former value and $q-1$ with the latter value.

The process will be the same for any number of sets of equal roots.

COR. 2. If $f(x)=0$ has m equal roots, $f'(x)=0$ has $m-1$ equal roots whose value is the same as that of one of the equal roots of the first equation.

If an equation is suspected of having equal roots, the highest common divisor of $f(x)$ and $f'(x)$ should be found. Suppose, for example, that the divisor is x^2-4x+4; the roots of $x^2-4x+4=0$, being 2 and 2, we have now $m-1$ of the m equal roots which the given equation $f(x)=0$ must have; hence 2, 2, 2 are roots of the equation, and since $f(x)$ is divisible by $(x-2)(x-2)(x-2)$, we may, by dividing, reduce the degree of the equation so that there is left for solution an equation of the $(n-3)$ degree.

EXAMPLES.

(NOTE. Before evaluating for any quantity, make as much use as possible of the foregoing propositions, including Descartes' Rule of Signs.)

I.

Find the roots of the following equations:

1. $x^3-3x^2-6x+8=0$.
2. $x^4-2x^2+1=0$.
3. $x^4+4x^3+x^2-8x-6=0$.
4. $x^5+x^4-8x^3-6x^2=0$.
5. $x^5-7x^4-3x^3+69x^2+4x-120=0$.

(The following have equal roots.)

6. $x^4-2x^3+3x^2+2x-4=0$.
7. $x^6-6x^5+3x^4+12x^3-9x^2-6x+5=0$.
8. $x^4-6x^3+6x^2+18x-27=0$.

II.

1. Find the condition that the equation

$$x^3+3ax^2+36x+c=0$$

may have two roots equal.

2. Find the condition that the equation

$$x^4+ax^2+b=0$$

may have equal roots, and determine whether it can have *three equal roots.*

III.

Show the general features of the graphs for equations of the following description:

1. An equation of the third degree, all of its roots real and two of them equal.

2. An equation formed by equating to zero the first derivative of the function in the preceding example.

3. An equation of the third degree, all of its roots equal and positive.

4. An equation of the fourth degree, all of its roots imaginary.

36. Thus far we have considered only such equations as have unity for the coefficient of the highest power of x. In case the coefficient of the highest power is not unity, the equation may be transformed into one in which the coefficient of the highest power is unity and the other coefficients are integral, the original coefficients being integral.

Suppose we have

$$p_0x^n + p_1x^{n-1} + p_2x^{n-2} + \cdots + p_{n-1}x + p_n = 0;$$

then $$x^n + \frac{p_1}{p_0}x^{n-1} + \frac{p_2}{p_0}x^{n-2} + \cdots + \frac{p_{n-1}}{p_0}x + \frac{p_n}{p_0} = 0.$$

Now let $x = \frac{y}{A}$, in which A is at first an undetermined quantity.

Substituting this value of x,

$$\left(\frac{y}{A}\right)^n + \frac{p_1}{p_0}\left(\frac{y}{A}\right)^{n-1} + \cdots + \frac{p_{n-1}}{p_0}\left(\frac{y}{A}\right) + \frac{p_n}{p_0} = 0.$$

Multiplying by A^n,

$$y^n + \frac{Ap_1}{p_0} y^{n-1} + \cdots + \frac{A^{n-1}p_{n-1}}{p_0} y + \frac{A^n p_n}{p_0} = 0.$$

If, now, A be taken equal to p_0, we have an equation of the form

$$y^n + P_1 y^{n-1} + P_2 y^{n-1} + \cdots + P_{n-1} y + P_n = 0,$$

in which the coefficients P_1, P_2, etc., are all integral. After solving this equation in y, the root-values of x in the original equation may be found from the relation $x = \frac{y}{A}$.

EXAMPLES.

Find the roots of the following equations:

1. $10x^3 - 17x^2 + x + 6 = 0.$
2. $3x^3 - 2x^2 - 6x + 4 = 0.$
3. $2x^3 - 3x^2 + 2x - 3 = 0.$
4. $x^4 - \frac{1}{2}x + \frac{3}{16} = 0.$

37. It is required to find the condition that one of the roots of the equation

$$p_0 x^n + p_1 x^{n-1} + \cdots + p_{n-1} x + p_n = 0 \qquad (1)$$

shall be infinitely great.

Put $x = \frac{1}{y}$; substituting this value of x in (1),

$$p_0\left(\frac{1}{y}\right)^n + p_1\left(\frac{1}{y}\right)^{n-1} + \cdots + p_{n-1}\left(\frac{1}{y}\right) + p_n = 0,$$

or

$$p_0 + p_1 y + \cdots + p_{n-1} y^{n-1} + p_n y^n = 0. \qquad (2)$$

If $p_0 = 0$, the equation may be written

$$y(p_1 + p_2 y + \cdots + p_{n-1} y^{n-2} + p_n y^{n-1}) = 0.$$

Hence 0 is one of the roots of the equation in y when $p_0 = 0$; but when $y = 0$, $x = \infty$. Hence *the condition that one of the roots of an equation shall be infinity is that the coefficient of the highest power of the unknown quantity shall be zero.* If both p_0 and p_1 are zero, two roots of equation (2) are zero; hence two roots of (1) are infinitely great.

(The principle of this article is of importance in the theory of asymptotes. See Williamson's *Differential Calculus*, Chapter XIII.)

Two equations are said to be *identical* if their corresponding coefficients are proportional.

Thus $p_0 x^n + p_1 x^{n-1} + \cdots + p_{n-1} x + p_n = 0$ (1)

and $P_0 x^n + P_1 x^{n-1} + \cdots + P_{n-1} x + P_n = 0$ (3)

are identical if

$$\frac{p_0}{P_0} = \frac{p_1}{P_1} = \cdots = \frac{p_{n-1}}{P_{n-1}} = \frac{p_n}{P_n},$$

and equation (3) might have been derived from (1) by multiplying the first term by $\frac{P_0}{p_0}$, the second by $\frac{P_1}{p_1}$, and so on.

38. Let

$$f(x) = (x - \alpha_1)(x - \alpha_2)(x - \alpha_3) \cdots (x - \alpha_n) = 0.$$

If the multiplication here indicated be performed, we have

$$x^n - (\alpha_1 + \alpha_2 + \cdots + \alpha_n) x^{n-1} + (\alpha_1\alpha_2 + \alpha_1\alpha_3 + \alpha_2\alpha_3 + \cdots) x^{n-1} + \cdots + (-1)^n (\alpha_1\alpha_2\alpha_3 \cdots \alpha_n) = 0.$$

It will be seen that the coefficient of x^n is $+1$; the coefficient of x^{n-1} is the sum of the roots, preceded by the minus sign; the coefficient of x^{n-2} is the sum of the product of the roots taken two and two; the coefficient of x^{n-3} is the sum of the product of the roots taken three and three, and preceded by the minus sign. The law of the formation of the coefficients is now apparent, and we have finally for the last term the product of all the roots, preceded by the minus sign if there is an odd number of roots. If this equation be identical with

$$x^n + p_1x^{n-1} + p_2x^{n-2} + \cdots + p_{n-1}x + p_n = 0,$$

by the preceding article we have

$$\begin{gathered} -(\alpha_1 + \alpha_2 + \cdots + \alpha_n) = p_1, \\ +(\alpha_1\alpha_2 + \alpha_1\alpha_3 + \alpha_2\alpha_3 + \cdots) = p_2, \\ \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ (-1)^n(\alpha_1\alpha_2\alpha_3 \cdots \alpha_n) = p_n; \end{gathered}$$

a set of n independent equations in the n unknown quantities $\alpha_1, \alpha_2, \cdots \alpha_n$, the known quantities being $p_1, p_2, \cdots p_n$.

Now it is obvious from the definition of a root, and in advance of investigation, that a root of an equation must be some function of the coefficients in that equation; and it would seem as if the above set of equations furnished us with material for determining *what* functions the roots are of the coefficients; that is, it would seem as if we might solve these n equations, obtaining the value of each root. A general solution would, of course, furnish formulas by means of which the roots could be determined in any particular case. Since the quantities $\alpha_1, \alpha_2, \alpha_3$, etc., are all involved in the same way in these *n equations,* there is no advantage in eliminating certain quantities rather than others. Suppose the $n-1$ quantities $\alpha_2, \alpha_3, \alpha_4, \cdots \alpha_n$, eliminated so that we have finally an equation in α_1. In whatever way the elimination is effected, this equation is of the nth degree in α_1; in fact, it presents the original function with α_1 in place of x; hence no progress is made toward the solution of the given equation in x.

39. The principles of the foregoing articles are of service in the determination of the roots of equations, but it has been seen that some of the principles apply only to equations having special characteristics, while others afford methods which are more or less tentative. We proceed to a theorem which enables us to determine the number and situation of the real roots of any numerical equation of the form

$$x^n + p_1 x^{n-1} + \cdots + p_{n-1} x + p_n = 0,$$

n being positive and integral, and the equation having no equal roots. This theorem, named from its discoverer, Sturm,* is as follows:

Given the equation $\mathrm{f}(\mathrm{x}) = 0$ *without equal roots. Let the operation of finding the highest common divisor of* $\mathrm{f}(\mathrm{x})$ *and* $\mathrm{f}'(\mathrm{x})$ *be performed with this modification, that the sign of every remainder is changed before it is used as a divisor. The given equation having no equal roots, we arrive at last at a numerical remainder not zero. Let the sign of this remainder also be changed. Let* $\mathrm{f}_1(\mathrm{x}), \mathrm{f}_2(\mathrm{x}), \cdots \mathrm{f}_{n-1}(\mathrm{x})$ *be the series of the modified remainders so that we have for*

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* Ce grand géomètre communiqua à l'Académie des Sciences, en 1829, la démonstration de son théorème qui constitue l'une des plus brillantes découvertes dont se soit enrichie l'Analyse mathématique. — *Serret.*

the entire series $f(x), f'(x), f_1(x), f_2(x), \cdots f_{n-1}(x)$, *the series decreasing in degree from the* n*th to the zero degree inclusive. Let* a *and* b (*where* $b > a$) *be any two real quantities; then the number of real roots of* $f(x) = 0$ *between* a *and* b *is equal to the excess of the number of variations in the series of signs of* $f(a)$, $f'(a)$, $f_1(a), \cdots f_{n-1}(a)$ *over the number of variations in the series of signs of* $f(b)$, $f'(b)$, $f_1(b), \cdots f_{n-1}(b)$.

$f_1(x)$, $f_2(x), \cdots f_{n-1}(x)$ may be called *Sturmian Functions.* Let $q_0, q_1, \cdots q_{n-2}$ denote the successive quotients which arise in the process of producing the Sturmian functions; then we have

$$f(x) = q_0 f'(x) - f_1(x),$$

$$f'(x) = q_1 f_1(x) - f_2(x),$$

$$f_1(x) = q_2 f_2(x) - f_3(x),$$

$$f_2(x) = q_3 f_3(x) - f_4(x),$$

$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$

$$f_{n-3}(x) = q_{n-2} f_{n-2}(x) - f_{n-1}(x).$$

From these relations two conclusions can be drawn:

First, two consecutive functions cannot vanish for the same value of x. Suppose the contrary to be true, and assume that $f_1(x)$ and $f_2(x)$ become equal to zero for the same value of x; then from the third of the above equations, $f_3(x)$ must also equal zero; and if $f_2(x)$ and $f_3(x)$ vanish simultaneously, $f_4(x)$ must also vanish in order that the fourth equation may be true. Finally, we shall have $f_{n-1}(x)$ vanishing with the rest; but this last Sturmian function, being a constant, cannot vanish for any value of x.

Second, when any intermediate function vanishes, the function preceding and the one succeeding have unlike signs. Suppose, for instance, that $f_3(x)$ vanishes for some value of x, as c. We have just seen that neither $f_2(x)$ nor $f_4(x)$ can vanish simultaneously with $f_3(x)$; hence the fourth equation of the above set becomes $f_2(x) = -f_3(x)$.

Now no alteration can be made in the series of signs of the functions

$$f(x), f'(x), f_1(x), \cdots f_{n-1}(x),$$

except when x passes through a value which makes some function vanish. We proceed to the proof of the following sub-propositions:

1. *No variation is lost or gained in consequence of* x *passing through a value which makes any function except* f(x) *vanish.*

2. *When* x *passes through a value that makes* f(x) *vanish, one variation, and only one, is lost.*

Let c be a value of x which makes some other function than $f(x)$ vanish. For example, suppose $f_3(c) = 0$. As neither $f_2(x)$ nor $f_4(x)$ vanishes when $x = c$, neither of them can change sign as x passes through c, but $f_3(x)$ will or will not change sign, according as the equation $f_3(x) = 0$ has an odd or even number of equal roots of the value c; hence the signs of $f_3(x)$ just before $x = c$ and just after $x = c$ may be any one of the four cases, $++$, $--$, $+-$, $-+$; hence if $c - h$ be the value of x just before $x = c$, and $c + h$ its value just after $x = c$, and if $f_2(x)$, $f_3(x)$, $f_4(x)$ be evaluated first for $c - h$, and then for $c+h$, all the possible arrangements of signs that can occur are:

$x = c - h$				$x = c + h$		
$f_2(c-h)$	$f_3(c-h)$	$f_4(c-h)$		$f_2(c+h)$	$f_3(c+h)$	$f_4(c+h)$
$+$	$+$	$-$		$+$	$+$	$-$
$+$	$-$	$-$		$+$	$-$	$-$
$+$	$+$	$-$		$+$	$-$	$-$
$+$	$-$	$-$		$+$	$+$	$-$
$-$	$+$	$+$		$-$	$+$	$+$
$-$	$-$	$+$		$-$	$-$	$+$
$-$	$+$	$+$		$-$	$-$	$+$
$-$	$-$	$+$		$-$	$+$	$+$

These signs should be read horizontally; it will be noticed that in every case there is one variation and one permanence.

Note. If $h = 0$, so that $f_3(c \mp h)$ is actually zero, the middle column of signs drops out, and we have simply $+ -$ or $- +$; *i.e.* one variation as before. Further, non-consecutive functions may vanish for the same value of x. When x reaches such a value, the number of signs in the series to be considered is at that instant diminished by the number of functions which vanish; but from the above argument it is seen that only *permanences* disappear, and that these reappear as soon as x has passed that value which causes the several non-consecutive functions to vanish.

We have finally to consider the effect of a passage of x *through a root* of $f(x) = 0$. Suppose α a root, so that

$f(\alpha)=0$. Let h be a positive quantity, so that $\alpha-h$ is the value of x just before the assumed root value is reached, and $\alpha+h$ is the value of x just after passing that value. By Art. 10,

$$f(\alpha-h)=f(\alpha)-hf'(\alpha)+\frac{h^2}{\underline{|2}}f''(\alpha)-\frac{h^3}{\underline{|3}}f'''(\alpha)+\cdots;$$

also

$$f(\alpha+h)=f(\alpha)+hf'(\alpha)+\frac{h^2}{\underline{|2}}f''(\alpha)+\frac{h^3}{\underline{|3}}f'''(\alpha)+\cdots.$$

Since $f(\alpha)=0$, these two equations may be written

$$\frac{f(\alpha-h)}{h}=-f'(\alpha)+\frac{h}{\underline{|2}}f''(\alpha)-\cdots$$

$$\frac{f(\alpha+h)}{h}=f'(\alpha)+\frac{h}{\underline{|2}}f''(\alpha)+\cdots,$$

or, if h be taken indefinitely small,

$$\frac{f(\alpha-h)}{h}=-f'(\alpha), \tag{1}$$

$$\frac{f(\alpha+h)}{h}=+f'(\alpha). \tag{2}$$

Now since the equation $f(x)=0$ has no equal roots, and consequently the equations $f(x)=0$ and $f'(x)=0$ have no common root, it follows that $f(x)$ changes sign when x passes through the value α, and $f'(x)$ does not change sign; *i.e.* the quantities $f(\alpha-h)$ and $f(\alpha+h)$ have unlike signs, whilst $f'(\alpha-h)$ and $f'(\alpha+h)$ have like signs. Summarizing the argument, we have

$$\begin{array}{ll} f(\alpha-h)\,\}\,\pm \qquad & f'(\alpha-h)\,\}\,\mp \\ f(\alpha+h)\,\}\,\mp \qquad & f'(\alpha+h)\,\}\,\mp \end{array}$$

Reading the upper signs of the functions
horizontal line, and then the upper signs in the
and reading the lower signs in the same mann
that in each case a variation is succeeded by
nence. Therefore if the entire series of func
$f'(x)$, $f_1(x)$, $f_2(x)$, etc., be evaluated for a
a is a root of $f(x)=0$ and h is indefinitely
if the series be then evaluated for $a+h$, t
series of signs must have one less variation th
series has; and since one variation must thus
to a permanence each time that x passes thro
of $f(x)=0$, and since no variation can be
passes through other values, the theorem is est

40. In order to find the whole number of re
an equation $f(x)=0$, we first make $x=-\infty$ i
of functions $f(x)$, $f'(x)$, $f_1(x)$, etc., and not
ber of variations; we then mak [redacted]
number of variations. The
variations in the first series
second series is the whole nu
is made equal to 0 in the seri
ber of variations when $x=-$
$x=0$ determines how many
negative.

It is to be understood that
functions, not only must the
changed before it is used as
factor must *not* be introduced
operation. (It is left to the
remainders *with their signs cha*
nary remainders, are taken for the Stu

Since $f(x)$ and $f'(x)$ must have unlik

$$\left.\begin{array}{l} \text{If } a_1 \text{ is } b_1, \\ \quad\text{or} \\ \text{If } a_2 \text{ is } b_2, \\ \quad\text{or} \\ \cdot \quad \cdot \quad \cdot \quad \cdot \\ \quad\text{or} \\ \text{If } a_n \text{ is } b_n, \end{array}\right\} c \text{ is } d; \tag{3}$$

[illegible] must have means for the elimination of all
[illegible]dents except one; the one that remains being
simple or compound, of *c*'s being *d* in the case
On the other hand, to affirm or assume the
[illegible] γ is δ, and thence infer the antecedent, α is
[illegible] does not necessarily involve a fallacy. This
form: if γ is δ, α is β, is evidently what is
[illegible] mathematics as the *converse;* as such it
[illegible]oof, although it may be observed that in
[illegible] if not in all, this proof is unnecessary pro-
the conclusion, γ is δ, adequately expresses
[illegible]lved in α's being β; we may then begin with
[illegible]remise, and make the other the conclusion.
[illegible]ple, suppose we have the proposition: If n
[illegible]rallel, they make equal angles (θ) with a
This cannot be converted into the proposi-
[illegible]ines make equal angles (θ) with a given line,
[illegible]allel. But the conclusion may be taken for
[illegible] the condition for conclusion, in the more
[illegible]ment: If n lines are parallel, they make
(θ, θ', θ'') with three given lines. In read-
[illegible]culus the student will find examples of incom-
[illegible]ns. Thus we have the theorem: If $f(x)$
[illegible], $f'(x)$ equals zero. It does not follow
[illegible] equals zero, $f(x)$ is necessarily a maxi-

mum. But suppose the theorem stated as follows: If $f(x)$ is a maximum when $x = a$, the first derivative which does not vanish in the development of $f(a \pm h)$ is an even derivative, and is negative. The converse of this proposition is true, and requires no proof; or more properly speaking, it is proved as soon as the direct proposition is proved. See Williamson's *Differential Calculus*, Art. 138.

Forms (1) and (2) may be avoided by employing the equivalent form:

$$\text{The case of} \left\{ \begin{matrix} \alpha\text{'s being } \beta \\ a\text{'s being } b \end{matrix} \right\} \text{is the case of} \left\{ \begin{matrix} \gamma\text{'s being } \delta. \\ c\text{'s being } d. \end{matrix} \right. \qquad (4)$$

Finally, (4) may be expressed,

$$\text{All } P \text{ is } Q. \qquad (5)$$

A proposition in form (5) is described as a ***universal affirmative.*** The subject is said to be ***distributed,*** whilst the predicate is, in general, not distributed. Thus

All freshmen are undergraduates

is a proposition referring to all freshmen, but not to all undergraduates. What we mean is, that all freshmen are some undergraduates. Evidently the subject cannot be taken for predicate, and predicate for subject, in another universal affirmative. We may only affirm that

Some undergraduates are freshmen;

thus converting a universal into a ***particular affirmative.*** The equivalence of forms (1), (4), (5) may be shown in an example as follows:

If an equation is of the third degree, it has at least one real root.

The case of an equation being of the third degree is *the case of* its having at least one real root.

All equations of the third degree have at least one real root.

Whenever all Q is referred to through all P, the proposition may be reversed, forming a new universal affirmative. If, then, we write a double form for (5), as

$$\text{All } P \text{ is } \begin{cases} \text{(some) } Q; \\ \text{(all) } Q; \end{cases} \tag{6}$$

we shall have the corresponding double form:

$$\left.\begin{array}{l} \text{Some } Q \\ \text{All } Q \end{array}\right\} \text{is } P. \tag{7}$$

Mathematics is chiefly concerned with propositions which conform to the second type under (6), *i.e.* propositions in which the predicate as well as the subject is distributed. Thus in Art. 35, the theorem is established by showing that if $f(x)$ has equal roots, the highest common divisor of $f(x)$ and $f'(x)$ is itself a function of x, and the argument involves the proposition: All equations having equal roots are all equations such that the highest common divisor of the function and its first derivative is a function of x: *i.e.* all P is all Q; and conversely, all Q is all P.

The student will find many universal affirmatives in the preceding pages, although they may not at first appear to be such. It is important to observe that a proposition of this description is under discussion whenever we employ the equation

$$f(x) = x^n + p_1x^{n-1} + \cdots + p_{n-1}x + p_n = 0;$$

for, by using the arbitrary constants n, p_1, p_2, $\cdots p_n$, one equation is made to include all equations of all finite degrees in *one* unknown quantity; and in establishing any truth in regard to this equation, that truth is established for all equations of the class embraced in the general equation.

The process by which we discover the conclusions necessarily implied or involved in given conditions is described as *deductive* reasoning. The most common variety of such reasoning is characterized by the presence of two categorical propositions called *premises*, one of which must be universal, and one of which must be affirmative; from these a conclusion, either universal or particular, is drawn, or is said to follow. As an example of this kind of reasoning, if we make the two affirmations:

All freshmen are undergraduates;
All freshmen study mathematics;

we are obliged to conclude that

Some undergraduates study mathematics.

Again, from the statements:

All logarithms are exponents;
.301030 is a logarithm;

it follows that

.301030 is an exponent.

As another illustration:

No determinants have an odd number of constituents;

$\begin{vmatrix} 2 & 0 \\ 1 & 3 \end{vmatrix}$ is a determinant;

Therefore $\begin{vmatrix} 2 & 0 \\ 1 & 3 \end{vmatrix}$ has not an odd number of constituents.

Here the first premise is a universal negative, the second premise a particular affirmative, whilst the conclusion is a particular negative. The student will easily add other combinations of premises to those given above. Thus a case might be given in which one premise should be a universal affirmative, and the other a particular negative.

Besides arguments composed entirely of simple categorical propositions, deductive reasoning includes certain other forms in which there appear categorical propositions united by a conjunction. Thus if we affirm:

$$\begin{aligned}&p \text{ is } q_1 \text{ or } q_2 \text{ or} \cdots \text{or } q_n;\\ &p \text{ is } q_1;\end{aligned} \tag{8}$$

we conclude that

$$p \text{ is not } q_2 \text{ or } q_3 \text{ or} \cdots \text{or } q_n.$$

Again, from the statements:

$$\begin{aligned}&p \text{ is } q_1 \text{ or } q_2 \text{ or} \cdots \text{or } q_n;\\ &p \text{ is not } q_2 \text{ or} \cdots \text{or } q_n;\end{aligned}$$

it follows that

$$p \text{ is } q_1.$$

An illustration will occur to those who have read Analytic Geometry: In the study of the equation

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0,$$

the function $h^2 - ab$ is formed. In any given equation of the second degree in x and y, $h^2 - ab$ is 0 or >0 or <0; suppose it is >0; then it is not 0 or <0.

The object in introducing in this connection any of these forms, whether categorical or not, is mainly to call attention to their constant occurrence in mathematics.

For a detailed and systematic examination of deductive reasoning, the student is referred to any of the standard text-books on the subject. Besides the disguising of the universal affirmative, already referred to, the frequent suppression, or non-expression, of a premise should be observed. These suppressed premises are, however, none the less present and essential to a sound argument. Finally, as regards the universal affirmatives of mathematics, the fundamental ones are the two axioms: 'Things equal to the same thing are equal to each other,' and 'The sums of equals are equal.' These are the *κοιναὶ ἔγγοιαι* which stand at the head of Euclid's list. The other so-called axioms which relate to pure quantity may be derived from these elementary truths.

Occasionally a method other than the deductive is followed in order to establish a mathematical proposition. Thus in Art. 25, instead of taking an equation of the *n*th degree as in most of the other articles, an equation of the fourth degree has been used. Any truth established respecting the equation of that article will be general so far as the coefficients are concerned, but it will be particular as regards the degree of the equation. But from the operation performed on the fourth degree equation the student perceives what the result would be in the case of equations of the fifth, sixth, and higher degrees. He performs the operation potentially on equations of those degrees also, and reaches a general result, *i.e.* a truth concerning the equation of the *n*th degree, without actually using that equation in the process. A similar generalizing occurs when, by examining the coefficients of the first few terms of the Binomial Theorem, we are able to express the law of their formation in the formula for the *n*th term. Reasoning of this kind has been called 'mathematical induction' because it bears a resemblance to some of the methods employed in inductive reasoning proper. The results of mathematical induction are general, but its material consists of particular truths. In some instances, mathematics presents problems which suggest methods partaking of the nature of experiment. A simple case occurs in connection with the graph of $x+2$, Art. 27. In that article it is stated that the graph is always a straight line when $f(x)$ is of the first degree, and the required graph was drawn accordingly; but independently of that statement, the student might have located a large number of graphic points, and then observed that the line connecting them was very nearly straight, becoming more so as the measurements were made more precise. The construction of the graphs of several special linear functions would doubtless have led to the conviction that *all* functions of the first degree have rectilinear graphs. But the universal proposition would not have been proven; the number of graphs constructed would have been insignificant compared with those left unconstructed, and no adequate reason could be assigned why the next graph should not be a curved line. On the other hand, if the properties of the graph of the general linear function $ax+b$ could be discovered from a study of the function itself, and if it could be proven deductively that the graph of $ax+b$ is a straight line, there would be no need to raise the question of the nature of the graph of any particular linear function. The peculiar power of Analytic or Co-ordinate Geometry consists in the fact that being algebraic it deals with universals, and is enabled to employ the methods of algebra to establish universal affirmatives which admit of a geometrical interpretation.